

Asymptotically locally flat spacetimes and dynamical black flowers in three dimensions

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Abstract

The theory of massive gravity proposed by Bergshoeff, Hohm and Townsend is considered in the special case of the pure irreducibly fourth order quadratic Lagrangian. It is shown that the asymptotically locally flat black holes of this theory can be consistently deformed to “black flowers” that are no longer spherically symmetric. Moreover, we construct radiating spacetimes settling down to these black flowers in the far future. The generic case can be shown to fit within a relaxed set of asymptotic conditions as compared to the ones of general relativity at null infinity, while the asymptotic symmetries remain the same. Conserved charges as surface integrals at null infinity are constructed following a covariant approach, and their algebra represents BMS_3 , but without central extensions. For solutions possessing an event horizon, we derive the first law of thermodynamics from these surface integrals.

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I. INTRODUCTION

In three spacetime dimensions, since the Riemann tensor can be expressed in terms of the Ricci tensor and the scalar curvature, general relativity in vacuum does not admit propagating degrees of freedom. Furthermore, black hole solutions in vacuum exist only in the case of negative cosmological constant [1, 2]. Thus, it is interesting to explore whether different simple models of three-dimensional gravity could capture further key properties of gravity in four dimensions. This seems to be the case for the theory of massive gravity proposed by Bergshoeff, Hohm and Townsend (BHT) [3, 4]. Hereafter, we consider this theory with the purely quadratic action given by

$$I[g] = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(R_{\mu\nu} R^{\mu\nu} - \frac{3}{8} R^2 \right). \quad (1)$$

The corresponding field equations

$$2\Box R_{\mu\nu} - \frac{1}{2}\nabla_\mu \nabla_\nu R - \frac{1}{2}\Box R g_{\mu\nu} + 4R_{\mu\sigma\nu\rho} R^{\sigma\rho} - \frac{3}{2}R R_{\mu\nu} - R_{\sigma\rho} R^{\sigma\rho} g_{\mu\nu} + \frac{3}{8}R^2 g_{\mu\nu} = 0, \quad (2)$$

are irreducibly of fourth order and propagate a single degree of freedom [4, 5]. Remarkably, these field equations admit a static asymptotically locally flat black hole solution, whose metric reads [6]

$$ds^2 = -(br - \mu) dt^2 + \frac{dr^2}{br - \mu} + r^2 d\phi^2. \quad (3)$$

This spacetime is conformally flat¹, and its Ricci scalar is given by

$$R = -\frac{2b}{r}, \quad (4)$$

so that it possesses a spacelike singularity at the origin, which is surrounded by an event horizon located at $r_+ = \mu/b$, provided $b > 0$ and $\mu > 0$. In the case of $\mu = 0$, there is a NUT on top of the null singularity. Note that its Hawking temperature, which turns out to be

$$T = \frac{b}{4\pi}, \quad (5)$$

depends only on one of the integration constants in (3). One of our purposes is to compute the mass of this black hole from a surface integral, which is not a simple task for a fourth-order theory with quadratic terms in the curvature.

¹ Indeed, this class of black holes was first found for conformal gravity in vacuum [7], whose field equations imply the vanishing of the Cotton tensor. For this reason, it has been shown that these black holes also solve the field equations of the Poincaré gauge theory [8].

In the next section, we show how the black hole solutions described by (3) can be deformed to “black flowers” that are no longer spherically symmetric. Their extension to time-dependent Robinson-Trautman-like solutions is also presented. In section III, the asymptotic conditions of general relativity at null infinity [10–12] are relaxed to accommodate these solutions and the asymptotic symmetries are worked out. In Section IV, conserved charges as surface integrals at null infinity are constructed following a covariant approach [13, 14]. Their algebra corresponds to BMS_3 without central extensions. The global charges of the (rotating) black holes as well as the ones of the dynamical black flowers are explicitly computed in Section V. Section VI is devoted to the derivation of the first law of thermodynamics from the surface integrals when event horizons are present.

II. DYNAMICAL BLACK FLOWERS

It is useful to express the black hole metric (3) in terms of a set of coordinates adapted to null infinity. This can easily be done by setting $u = t - r^*$, where r^* stands for the tortoise coordinate defined through $dr^* = \frac{dr}{br - \mu}$, so that the black hole metric read

$$ds^2 = -(br - \mu) du^2 - 2dudr + r^2 d\phi^2 . \quad (6)$$

These metrics can be deformed along the spacelike Killing vector ∂_ϕ , yielding

$$ds^2 = -(br - \mu) du^2 - 2dudr + (r - \mathcal{H}(u, \phi))^2 d\phi^2 , \quad (7)$$

where $\mathcal{H}(u, \phi)$ is periodic in the angular coordinate ϕ . Even though the deformation is not of Kerr-Schild type, the field equations (2) surprisingly reduce to a single linear differential equation given by

$$\partial_u \left(\partial_u \mathcal{H} + \frac{b}{2} \mathcal{H} \right) = 0 . \quad (8)$$

The general solution reads

$$\mathcal{H}(u, \phi) = \mathcal{A}(\phi) + \mathcal{B}(\phi) e^{-\frac{b}{2}u} , \quad (9)$$

where $\mathcal{A}(\phi)$ and $\mathcal{B}(\phi)$ are arbitrary periodic functions. Note that the deformed metrics are still conformally flat. Their Ricci scalar is

$$R = \frac{2b}{r - \mathcal{H}} , \quad (10)$$

signaling the existence of a curvature singularity at $r = \mathcal{H}(u, \phi)$, so that the range of the radial coordinate can be chosen as $\mathcal{H} < r < \infty$.

This class of solutions can be divided as follows: when $\mathcal{B}(\phi) = 0$, the resulting metrics are static while $\mathcal{B}(\phi) \neq 0$ leads to metrics describing dynamical spacetimes.

In the static case, the metric has an event horizon located at $r_+ = \mu/b$. In order for this horizon to enclose the curvature singularity, the function $\mathcal{A}(\phi)$ has to be bounded from above according to

$$\mathcal{A}(\phi) < r_+ . \quad (11)$$

The induced metric on a constant u slice of the horizon is given by

$$ds_h^2 = (r_+ - \mathcal{A}(\phi))^2 d\phi^2 , \quad (12)$$

and since the function $\mathcal{A}(\phi)$ is arbitrary, the horizon is generically not spherically symmetric. In particular, these spacetimes can describe a black flower. The associated temperature can be evaluated using the surface gravity and is found to coincide with the one of the spherically symmetric black holes given in (5).

In the dynamical case, the deformation $\mathcal{H}(u, \phi)$ propagates along outgoing null rays. At late retarded time, u going to infinity, this generic configuration tends to a static black flower. Therefore, one may view these solutions as describing a black flower surrounded by an outgoing graviton. This can be easily seen introducing Kruskal-Szekeres coordinates

$$U = -e^{-\frac{b}{2}u} ; \quad V = e^{\frac{b}{2}v} ; \quad r(U, V) = \frac{1}{b}(\mu - UV) \quad (13)$$

where $v = t + r^*$ is the advanced time. The metric then takes the form

$$ds^2 = -\frac{4}{b^2}dUdV + (r(U, V) - \mathcal{H}(U, V, \phi))^2 d\phi^2 , \quad (14)$$

with

$$\mathcal{H}(U, V, \phi) = \mathcal{A}(\phi) - \mathcal{B}(\phi)U . \quad (15)$$

This indicates that the spacetime possesses a horizon at $U = 0$ with a curvature singularity located at $r(U, V) = \mathcal{H}(U, V, \phi)$, i.e.

$$U(V - b\mathcal{B}(\phi)) = \mu - b\mathcal{A}(\phi) . \quad (16)$$

The future curvature singularity is surrounded by the horizon only when eq. (11) holds. On this horizon the metric reduces to the one of the static black flower while outside of the horizon, for $U < 0$, the spacetime is filled with outgoing radiation.

These solutions are similar to the Robinson-Trautman spacetimes found in four-dimensional Einstein gravity [15, 16], which describe Schwarzschild black holes surrounded by outgoing radiation. The main difference is that the time evolution of these four-dimensional analogues is described by a complicated fourth order nonlinear differential equation.

It is worth mentioning that deformations of exact solutions that are not of the standard Kerr-Schild type (see e.g. [17, 18]), have been found for different cases of BHT massive gravity theory [19], and also for Einstein-Gauss-Bonnet [20, 21] and Lovelock theories in higher dimensions [22].

III. RELAXED FALL-OFF CONDITIONS AND ASYMPTOTIC SYMMETRIES

Recently, there has been renewed interest in asymptotically flat spacetimes at null infinity in three and four dimensions (see, e.g., [12, 23–34]). Since the dynamical black flowers contain outgoing gravitational radiation, it is natural to study their asymptotic structure at null infinity. The metric (7) describing these dynamical black flowers can be shown to fit within a set of asymptotic conditions that are relaxed as compared to the standard ones for general relativity [10, 11]. Nevertheless, the asymptotic symmetry algebra remains the same. The suitably relaxed asymptotic conditions for the metric are

$$\Delta g_{AB} = h_{AB}r + \mathcal{O}(1) ; \Delta g_{rr} = \mathcal{O}(r^{-3}) ; \Delta g_{Ar} = \mathcal{O}(r^{-1}) , \quad (17)$$

where the functions h_{AB} depend only on $x^A = (u, \phi)$ and $\Delta g_{\mu\nu} = g_{\mu\nu} - \bar{g}_{\mu\nu}$ correspond to deviations from the Minkowski metric,

$$d\bar{s}^2 = -du^2 - 2dudr + r^2d\phi^2 . \quad (18)$$

The asymptotic conditions (17) are mapped into themselves under diffeomorphisms of the form,

$$\begin{aligned}\xi^u &= T(\phi) + u\partial_\phi Y(\phi) + \mathcal{O}(r^{-1}) , \\ \xi^r &= -r\partial_\phi Y(\phi) + \mathcal{O}(1) , \\ \xi^\phi &= Y(\phi) - \frac{1}{r}\partial_\phi(T(\phi) + u\partial_\phi Y(\phi)) + \mathcal{O}(r^{-2}) .\end{aligned}\tag{19}$$

The sub-space of vectors with vanishing $T(\phi), Y(\phi)$ form an ideal and the quotient algebra is BMS_3 , i.e., the semi-direct sum of the vector fields on the circle with the abelian ideal of supertranslations. Thus, remarkably, the relaxed behavior of the metric at infinity does not spoil the asymptotic symmetries obtained in Refs. [10, 11] for general relativity in three dimensions.

IV. CONSERVED CHARGES AT NULL INFINITY

In order to obtain the conserved charges associated to the asymptotic symmetries spanned by (19), it is useful to write the action (1) in terms of an auxiliary field $\ell_{\mu\nu}$, so that the field equations reduce from fourth to second order [3–5]. The action then reads

$$I[g, \ell] = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left(\ell^{\mu\nu} G_{\mu\nu} - \frac{1}{4} (\ell^{\mu\nu} \ell_{\mu\nu} - \ell^2) \right) ,\tag{20}$$

and the algebraic field equations associated to $\ell_{\mu\nu}$ are given by

$$G^{\mu\nu} - \frac{1}{2} (\ell^{\mu\nu} - g^{\mu\nu} \ell) = 0 .\tag{21}$$

On-shell, the auxiliary field turns out to be proportional to the Schouten tensor,

$$\ell_{\mu\nu} = 2 \left(R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \right) .\tag{22}$$

Varying the action with respect to the metric also produces second order field equations

$$\begin{aligned}\nabla^\alpha \nabla_\alpha \ell^{\mu\nu} - 2\nabla_\rho \nabla^{(\mu} \ell^{\nu)\rho} + \nabla^\mu \nabla^\nu \ell + g^{\mu\nu} (\nabla_\rho \nabla_\lambda \ell^{\rho\lambda} - \nabla^\alpha \nabla_\alpha \ell) + 4\ell^{\lambda(\mu} G_{\lambda}^{\nu)} \\ + \ell^{\mu\nu} R - \ell R^{\mu\nu} - g^{\mu\nu} \ell^{\rho\sigma} G_{\rho\sigma} + \ell^{\lambda\mu} \ell_{\lambda}^{\nu} - \ell \ell^{\mu\nu} - \frac{1}{4} g^{\mu\nu} (\ell_{\alpha\beta} \ell^{\alpha\beta} - \ell^2) = 0 .\end{aligned}\tag{23}$$

Following the covariant approach described in [13, 14], the conserved charges are given by,

$$Q_\xi = \int_0^1 ds \left(\frac{1}{2} \int_{\partial\Sigma} \varepsilon_{\nu\mu\alpha} \tilde{k}_\xi^{[\nu\mu]} dx^\alpha \right) .\tag{24}$$

In our case, the superpotential acquires the form

$$\begin{aligned} \left(\frac{8\pi G}{\sqrt{-g}} \right) \tilde{k}_\xi^{[\nu\mu]} &= P^{\mu\rho(\lambda\nu)\alpha\beta} \left(\frac{2}{3} \xi_\rho \nabla_\lambda k_{\alpha\beta} - \frac{1}{3} k_{\alpha\beta} \nabla_\lambda \xi_\rho \right) - T^{\mu\rho\lambda\nu\alpha\beta\delta\gamma} \left(\frac{1}{3} \xi_\rho h_{\alpha\beta} \nabla_\lambda \ell_{\delta\gamma} \right) \\ &+ U^{\mu\rho(\lambda\nu)\alpha\beta\delta\gamma} \left(\frac{2}{3} \xi_\rho \nabla_\lambda h_{\alpha\beta} - \frac{1}{3} h_{\alpha\beta} \nabla_\lambda \xi_\rho \right) \ell_{\delta\gamma} - (\mu \leftrightarrow \nu) , \end{aligned} \quad (25)$$

with

$$P^{\mu\nu\kappa\sigma\alpha\beta} = g^{\kappa\sigma} g^{\mu[(\alpha} g^{\nu)]\beta} + g^{\nu\sigma} g^{\mu[\kappa} g^{(\alpha]\beta)} + g^{\kappa(\alpha} g^{\mu[\nu} g^{\sigma]\beta)} , \quad (26)$$

$$U^{\mu\rho(\lambda\nu)\alpha\beta\delta\gamma} = \hat{Y}_{\eta\varepsilon}^{(\lambda\nu)\alpha\beta} P^{\mu\rho\gamma\delta\eta\varepsilon} - 2P^{\mu\rho(\lambda|\eta\varepsilon\gamma} \hat{X}_{\eta\varepsilon}^{\delta|\nu)\alpha\beta} + P^{\mu\delta(\lambda\nu)\alpha\beta} \bar{g}^{\gamma\rho} , \quad (27)$$

$$T^{\mu\rho\lambda\nu\alpha\beta\delta\gamma} = U^{\mu\rho(\lambda\nu)\alpha\beta\delta\gamma} + \frac{3}{2} \left(P^{\mu\rho\varepsilon\sigma\gamma} \hat{X}_{\sigma\varepsilon}^{\lambda\nu\alpha\beta} + 2 \left(P^{\mu\rho\sigma\lambda\varepsilon\gamma} + P^{\mu\rho\lambda\sigma\varepsilon\gamma} \right) \hat{X}_{\sigma\varepsilon}^{\delta\nu\alpha\beta} \right) , \quad (28)$$

and

$$\begin{aligned} P_{II}^{\mu\nu\delta\lambda\alpha\beta} &= \left(g^{\mu(\lambda} g^{\delta)(\beta} - \frac{1}{2} g^{\delta\lambda} g^{\mu(\beta} \right) g^{|\alpha)\nu} + \left(g^{\mu(\alpha} \bar{g}^{\beta)(\delta} - \frac{1}{2} g^{\alpha\beta} g^{\mu(\delta} \right) g^{|\lambda)\nu} \\ &- \frac{1}{2} g^{\mu\nu} \left(g^{(\delta(\alpha} g^{\beta)\lambda)} - \frac{1}{2} g^{\delta\lambda} g^{\alpha\beta} \right) , \end{aligned} \quad (29)$$

$$\hat{Y}_{\eta\varepsilon}^{\kappa\sigma\alpha\beta} = \frac{1}{2} \left(-\delta_{(\eta}^{\alpha} \delta_{\varepsilon)}^{\beta} g^{\sigma\kappa} - \delta_{(\eta}^{\kappa} \delta_{\varepsilon)}^{\sigma} g^{\alpha\beta} + 2\delta_{(\eta}^{\sigma} \delta_{\varepsilon)}^{(\alpha} g^{\beta)\kappa} \right) , \quad (30)$$

$$\hat{X}_{\sigma\beta}^{\lambda\eta\gamma\rho} = \frac{1}{2} \bar{g}^{\lambda\xi} \left(2\delta_{(\sigma}^{\eta} \delta_{\beta)}^{(\gamma} \delta_{\xi}^{\rho)} - \delta_{\xi}^{\eta} \delta_{\sigma}^{(\gamma} \delta_{\beta)}^{\rho)} \right) . \quad (31)$$

Here, it is assumed that $g_{\mu\nu} := g_{\mu\nu}^s$ and $\ell_{\mu\nu} := \ell_{\mu\nu}^s$ are one-parameter families of solutions, interpolating between the background fields $\bar{g}_{\mu\nu} = g_{\mu\nu}^0$, $\bar{\ell}_{\mu\nu} = \ell_{\mu\nu}^0$ and some given solution $g_{\mu\nu} = g_{\mu\nu}^1$ and $\ell_{\mu\nu} = \ell_{\mu\nu}^1$. The deviations $h_{\mu\nu}$ and $k_{\mu\nu}$ are defined as the tangent vectors to $g_{\mu\nu}^s$ and $\ell_{\mu\nu}^s$ in the solution space, i.e.,

$$h_{\mu\nu} = \frac{d}{ds} g_{\mu\nu}^s ; \quad k_{\mu\nu} = \frac{d}{ds} \ell_{\mu\nu}^s . \quad (32)$$

As a first check, for the static black hole (6) the interpolating metric $g_{\mu\nu}^s$ can be chosen as

$$g_{\mu\nu}^s \rightarrow ds_s^2 = -(1 + s(br - \mu - 1))du^2 - 2dudr + r^2 d\phi^2 , \quad (33)$$

and $\ell_{\mu\nu}^s$ is obtained plugging (33) in eq. (22). In particular, for the mass, the surface integral gives

$$M = Q(\partial_u) = \frac{b^2}{32G} . \quad (34)$$

Using an arbitrary set of interpolating metrics satisfying the asymptotic conditions given in eq. (17), the conserved charges (24) simplify to

$$Q_\xi = Q[T, Y] = \frac{1}{64\pi G} \int_0^{2\pi} d\phi \left((T + u\partial_\phi Y) h_{uu}^2 + 2Y h_{u\phi} h_{uu} + 4h_{uu} \partial_\phi Y + 4Y \partial_u h_{u\phi} \right) . \quad (35)$$

The black hole mass in (34) can then also directly be computed from (35), by taking into account that for the metric (6), the only nonvanishing deviation from the background is given by $h_{uu} = -b$.

Using the leading terms of the following asymptotic field equations

$$E_{uu} = -\frac{1}{4} (\partial_u h_{uu}^2) r^{-1} + \mathcal{O}(r^{-2}) = 0 , \quad (36)$$

$$E_{u\phi} = \frac{1}{4} (\partial_\phi h_{uu}^2 - 2\partial_u (h_{uu} h_{u\phi}) + 4\partial_u (\partial_\phi h_{uu} - \partial_u h_{u\phi})) r^{-1} + \mathcal{O}(r^{-2}) = 0 , \quad (37)$$

and the action of the asymptotic symmetries on the relevant dynamical fields

$$\delta_\xi h_{uu} = (T(\phi) + u\partial_\phi Y(\phi)) \partial_u h_{uu} + \partial_\phi (h_{uu} Y) , \quad (38)$$

$$\delta_\xi h_{u\phi} = h_{uu} \partial_\phi (T(\phi) + u\partial_\phi Y(\phi)) + (T(\phi) + u\partial_\phi Y(\phi)) \partial_u h_{u\phi} + \partial_\phi (h_{u\phi} Y) , \quad (39)$$

it is simple to verify that the algebra $\{Q_{\xi_1}, Q_{\xi_2}\} \equiv \delta_{\xi_2} Q_{\xi_1}$ realises the BMS₃ without central extension,

$$\{Q_{\xi_1}, Q_{\xi_2}\} \approx Q_{[\xi_1, \xi_2]} . \quad (40)$$

Expanding the charges in Fourier modes

$$\mathcal{P}_n = Q[e^{in\phi}, 0] ; \quad \mathcal{J}_n = Q[0, e^{in\phi}] , \quad (41)$$

the algebra takes the familiar form

$$\begin{aligned} i \{ \mathcal{J}_m, \mathcal{J}_n \} &= (m - n) \mathcal{J}_{m+n} , \\ i \{ \mathcal{J}_m, \mathcal{P}_n \} &= (m - n) \mathcal{P}_{m+n} , \\ i \{ \mathcal{P}_m, \mathcal{P}_n \} &= 0 . \end{aligned} \quad (42)$$

The vanishing of the potential central extensions of the algebra (42) can also be obtained by taking the flat limit of the extended two-dimensional conformal algebra given in [6], and also in [36, 37] for BHT massive gravity in the special case (with a unique maximally symmetric vacuum) from a holographic approach.

V. CONSERVED CHARGES FOR ROTATING BLACK HOLES AND DYNAMICAL BLACK FLOWERS

The theory under consideration also admits a rotating asymptotically locally flat black hole solution [6, 35, 36]. In outgoing null coordinates, the metric takes the form

$$ds^2 = -NFdu^2 - 2N^{\frac{1}{2}}dudr + (r^2 + r_0^2) (d\phi + N^\phi du)^2 , \quad (43)$$

where

$$F = br - \mu ; \quad N = \frac{(8r + a^2b)^2}{64(r^2 + r_0^2)} ; \quad N^\phi = -\frac{a}{2} \left(\frac{br - \mu}{r^2 + r_0^2} \right) ; \quad r_0^2 = \frac{a^2}{4} \left(\mu + \frac{a^2b^2}{16} \right) .$$

Note that when the angular parameter a vanishes, the solution reduces to the static black hole given in (6). Since the relevant deviations from the background are given by $h_{uu} = -b$ and $h_{u\phi} = -\frac{a}{2}b$, the mass and the angular momentum can be directly obtained using (35). Thus, one finds

$$M = Q(\partial_u) = \frac{b^2}{32G} , \quad (44)$$

$$J = Q(\partial_\phi) = \left(\frac{b^2}{32G} \right) a = Ma , \quad (45)$$

which is in agreement with the vanishing cosmological constant limit of BHT massive gravity in the special case that admits a unique maximally symmetric vacuum [36, 37]. Furthermore, one deduces that the only nonvanishing global charges associated with the rotating black hole are the mass and the angular momentum, and therefore, the integration constant μ turns out to be a “gravitational hair” parameter².

In the case of the dynamical black flowers (7), (9), the deformation with respect to the static black hole does not modify the values of h_{uu} and $h_{u\phi}$. Therefore, the charges remain the same, that is $M = \frac{b^2}{32G}$ and $J = 0$. Surprisingly, even though we have outgoing radiation the total mass at null infinity is constant, i.e., there is no “news” [38, 39]. When restricting to static black flowers, it is worth noticing that both μ and $\mathcal{A}(\phi)$ are hair parameters. In this sense, from the mode expansion of $\mathcal{A}(\phi)$, one might interpret static black flowers as black hole solutions with an infinite number of purely gravitational hair parameters.

² Curiously, if one compares these results with the ones in [6, 35, 36] in the presence of cosmological constant, the roles that the integration constants b and μ play in the global charges become interchanged.

VI. THERMODYNAMICS

The conserved charges (24) are also useful in the presence of horizons. Indeed, for a black hole with a Killing horizon, as for the static black flowers, one can use surface charges for $\xi = \partial_u$ to derive thermodynamical properties of the black holes as in [40, 41].

The conservation of the superpotential $\tilde{k}_\xi^{[\mu\nu]}$ associated to a Killing vector ξ implies

$$\delta \left(Q_\xi|_{r=\infty} + Q_\xi|_{r=r_+} \right) = 0 . \quad (46)$$

Remarkably, for the static black flowers the superpotential can be computed exactly, and is simply given by

$$\tilde{k}_\xi^{[ur]} = \frac{b\delta b}{32\pi G} . \quad (47)$$

As seen above, evaluating the charge at infinity one obtains the total energy, i.e.,

$$Q_\xi|_{r=\infty} = \frac{b^2}{32G} = M , \quad (48)$$

while the variation of the surface integral at the horizon $r = r_+$ gives

$$\delta Q_\xi|_{r=r_+} = -\frac{b\delta b}{16G} = -T\delta S . \quad (49)$$

The conservation law (46) then reduces to the first law of thermodynamics $\delta\mathcal{F} = 0$, where $\mathcal{F} = M - TS$ stands for the free energy. Hence, the black hole entropy can be expressed in terms of the surface integral

$$S = -2\pi \int_0^1 ds \left(\int_{\Sigma_h} \hat{\varepsilon}_{\mu\nu} \tilde{k}_\xi^{[\nu\mu]} d\phi \right) = \frac{\pi b}{4G}, \quad (50)$$

where $\hat{\varepsilon}_{\mu\nu}$ is the binormal to the bifurcation surface Σ_h , normalized as $\hat{\varepsilon}_{\mu\nu}\hat{\varepsilon}^{\mu\nu} = -2$. It is not surprising that this result differs from a quarter of the area of the event horizon since we are not dealing with general relativity.

As a final remark, it is worth pointing out that the results obtained here can also be generalized to the case of asymptotically locally flat hairy black holes with angular momentum (43). Curiously, this black hole has vanishing angular velocity for the horizon, i.e., $\Omega_+ = 0$ so that the first law reduces to

$$dM = TdS - \Omega_+ dJ = TdS . \quad (51)$$

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